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Fractional Differential Equations: Solution via the Homotopy Analysis Method

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Abstract

Fractional differential equations (FDEs) provide an effective mathematical framework for describing phenomena that evolve in both time and space and exhibit memory and hereditary effects. Their applications span a wide range of fields, including control theory, polymer mechanics, bioengineering, and other areas of applied science and engineering. The Homotopy Analysis Method (HAM) is a powerful analytical technique that constructs the solution in the form of an infinite series without requiring linearization, perturbation assumptions, or small parameters. HAM can be extended to fractional-order models involving Riemann–Liouville or Caputo fractional derivatives, leading to a comprehensive formulation of the method and providing valuable insight into the qualitative behaviour of the solutions. The present study documents the detailed application of HAM to nonlinear FDEs, with the objectives of demonstrating its analytical capability, establishing theoretical results concerning the existence and uniqueness of solutions, and investigating convergence behaviour together with error estimation. The analysis is restricted to nonlinear time-fractional differential equations involving fractional derivatives with respect to time only. A key feature of HAM is the freedom to choose the auxiliary linear operator. Although a linear counterpart is often selected to simplify the governing system, formulations based on Riemann–Liouville derivatives may become mathematically cumbersome. To avoid this difficulty, the proposed approach chooses the linear operator to be zero, which significantly simplifies the implementation while preserving the generality of the method. The nonlinear FDE is retained in its general form to highlight the flexibility of HAM and its capability to handle highly nonlinear problems, and appropriate boundary conditions are explicitly imposed to ensure a well-defined solution procedure. The formulation assumes Caputo fractional derivatives in the nonlinear term, consistent with the *time-fractional integration operator appearing on the left-hand side of the equation (Mohammed & Khlaif, 2014).*

Keywords: Fractional differential equations; Homotopy Analysis Method (HAM); Caputo fractional derivative; Nonlinear time-fractional equations; Convergence and error analysis.

1. Introduction

Fractional differential equations (FDEs) involve derivatives of non-integer order and provide a natural extension of classical time and space derivatives. They have become an important mathematical tool for modeling complex physical phenomena that exhibit memory, hereditary, and nonlocal effects. Such equations arise in a wide range of applications, including viscoelasticity, control systems, signal processing, diffusion processes, fluid mechanics, polymer dynamics, and biological systems. Because many of these models cannot be solved effectively by conventional analytical techniques, the development of reliable approximation methods has received considerable attention.

The Homotopy Analysis Method (HAM) is a powerful semi-analytical technique that constructs the solution of a nonlinear problem in the form of a convergent infinite series. Unlike perturbation methods, HAM does not require the presence of a small parameter, nor does it rely on linearization assumptions. The method provides greater flexibility in controlling convergence through the introduction of an auxiliary linear operator and a convergence-control parameter. As a result, HAM often determines solution behaviour and convergence properties more reliably than many comparable analytical approaches. The resulting series solution is expressed in an explicit power-series form, which can be evaluated efficiently to obtain accurate numerical approximations for FDEs subject to prescribed initial or boundary conditions. Furthermore, existence and uniqueness results can be established for the series solutions of a broad class of linear and nonlinear fractional differential equations.

The theoretical foundation of HAM is closely connected with the broader development of fractional calculus. In many physical systems, classical integer-order derivatives become inadequate in the presence of singularities, discontinuities, anomalous diffusion, and other nonlocal phenomena. This limitation motivates the extension of ordinary differentiation and integration to fractional-order operators. Among the various definitions proposed in the literature, the Riemann–Liouville and Caputo fractional derivatives are the most widely used. The Caputo derivative is particularly attractive for continuous-time models involving memory and delay effects because it allows the use of classical initial conditions expressed in terms of integer-order derivatives.

HAM has been successfully applied to a variety of fractional models, including inhomogeneous linear FDEs, viscoelastic models, biological systems, fractional low-pass filtering problems, string and wave equations, and fractional equations solved in combination with Laplace-transform techniques. In addition, finite-point and transform-based formulations have been developed to improve computational efficiency and reduce the complexity associated with relaxation-time representations and Fourier-based stability analyses.

Fractional calculus is now recognized as an essential framework for describing complex dynamical systems whose behaviour cannot be captured adequately by classical integer-order models. Modern fractional models may include space–time fractional operators, integro-partial differential equations, delay terms, distributed-order derivatives, and multi-term variable-order structures. These formulations provide a more transparent description of spatial and temporal interactions while preserving the essential physical characteristics of the underlying system.

The primary objective of the present study is to develop a deeper analytical and computational understanding of nonlinear time-fractional differential equations using HAM. The method is formulated in a sufficiently general setting so that it can accommodate higher-order, multi-term, and variable-order fractional models. Depending on the structure of the governing equations, the resulting systems may be solved completely in analytic form or treated semi-exactly to obtain highly accurate approximate configurations.

Previous implementations of HAM for initial- and boundary-value fractional problems have typically employed auxiliary functions or linear operators that were either fixed in advance or chosen to satisfy specific constraints. In contrast, the present formulation emphasizes the flexibility of the homotopy construction and the freedom in selecting the auxiliary components, which allows the method to be adapted to a much larger class of nonlinear fractional systems without altering the underlying physical model. This feature enhances both the mathematical generality and the practical applicability of the proposed approach for problems involving coupled time–space dynamics, memory effects, and nonlocal interactions (Mohammed & Khlaif, 2014).

2. Preliminaries

Fractional differential equations (FDEs) and mathematical models involving fractional-order derivatives arise naturally in many areas of science and engineering and have become a subject of significant research interest. Their importance stems from the ability of fractional operators to describe memory, hereditary, and nonlocal effects more accurately than classical integer-order derivatives. As a result, fractional models have been successfully applied in fields such as viscoelasticity, fluid mechanics, control theory, electrical circuits, diffusion processes, bioengineering, and mathematical biology.

To solve these equations, a wide variety of analytical and numerical techniques have been developed, including transform methods, decomposition methods, spectral approaches, finite difference schemes, and homotopy-based techniques. Among these, the Homotopy Analysis Method (HAM) introduced by Liao has emerged as one of the most effective and widely used approaches for nonlinear problems (Ahmad & Tauseef Mohyud-Din, 2014). HAM is particularly attractive because it is relatively easy to implement, does not require linearization or perturbation assumptions, and provides a systematic procedure for constructing convergent series solutions.

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The growing interest in fractional calculus is motivated by both its rich mathematical structure and its broad range of practical applications. Fractional differential and integral equations often provide more realistic models of complex dynamical systems than their integer-order counterparts. Consequently, considerable effort has been devoted to the development of reliable solution strategies capable of handling the strong nonlinearities and nonlocal characteristics that frequently appear in such models.

In this context, HAM offers several important advantages. The method introduces an auxiliary linear operator, an initial approximation, and a convergence-control parameter, which together provide substantial flexibility in controlling the convergence region and convergence rate of the resulting solution series. This flexibility allows HAM to produce highly accurate approximate solutions for a wide class of linear and nonlinear fractional differential equations, including equations with delay terms, variable coefficients, and coupled fractional operators. Because of these features, HAM has become a powerful analytical tool for investigating the qualitative and quantitative behaviour of fractional dynamical systems and continues to play an important role in the advancement of fractional-order modeling and analysis.

3. Construction of the Homotopy

The fractional differential equation (FDE) considered in this study is formulated as a nonlinear initial-value problem (H. Mohammed & I. Khlaif, 2014). The governing equation is subject to prescribed initial conditions, and the fractional derivative of order α is defined in the Caputo sense, which is particularly suitable for physical models because it allows the incorporation of classical initial conditions expressed in terms of integer-order derivatives.

Within the framework of the Homotopy Analysis Method (HAM), an auxiliary linear operator $\mathcal{L}$ is introduced to construct the homotopy and facilitate the derivation of the deformation equations. The embedding parameter q is allowed to vary continuously in the interval [0,1], and in some generalized formulations it may be extended to complex values in order to investigate analytic continuation and convergence properties. The unknown function $\phi(t;q)$ depends on both the independent variable $t$and the embedding parameter q, while the convergence-control parameter $h$plays a crucial role in determining the convergence region and convergence rate of the resulting homotopy series.

Approximate analytical solutions are obtained by expressing the nonlinear problem in a homotopy form and assuming an appropriate representation of the nonlinear term. The construction begins with the zeroth-order deformation equation, which establishes a continuous deformation from an initial approximation to the exact solution of the original nonlinear FDE (Ahmad & Tauseef Mohyud-Din, 2014). The corresponding homogeneous equation is then used to define the associated linear operator for the first-order Caputo fractional differential equation, enabling the recursive computation of higher-order approximation terms.

This formulation provides a systematic procedure for generating convergent series solutions and offers significant flexibility in handling nonlinear fractional models. By properly selecting the auxiliary linear operator and the convergence-control parameter, the HAM framework can produce highly accurate approximate solutions while maintaining analytical tractability for a broad class of nonlinear fractional initial-value problems.

4. Series Solution and Convergence

The series solution obtained through the Homotopy Analysis Method (HAM) is constructed from the zeroth-order deformation equation and represents a convergent approximation to the solution of the nonlinear fractional differential equation. Accordingly, the solution can be expressed in the form

$$u(t) \sim \sum_{n=0}^{\infty} u_n(t), \qquad (4.1)$$

where $u_n(t)$denotes the nth-order approximation term of the homotopy series.

The coefficients of the series are generated recursively through the relation

$$u_n(t) = \frac{h^n}{n!} \hat{L}^n[v_0(t) - u_0(t)], n = 0,1,2, \ldots, \qquad (4.2)$$

where $\hat{L}$ is the auxiliary linear operator, $u_0(t)$ is the initial approximation, $v_0(t)$ is the auxiliary function associated with the deformation process, and $h$is the convergence-control parameter that plays a central role in regulating the convergence behavior of the HAM series.

Following the extended notation introduced by Turkyilmazoglu, M. (2010), the fractional order $\alpha$of the governing FDE and the approximation order $n$may take values in the semi-closed interval

$$0 < \alpha \leq \frac{5}{4}, n = 0,1,2, \ldots.$$

Explicit expressions for the first-order and second-order approximations can be derived for arbitrary values of α, providing a practical basis for constructing higher-order analytical approximations.

The convergence of the HAM series depends strongly on both the choice of the auxiliary linear operator $\hat{L}$ and the convergence-control parameter h. For an appropriately selected operator, it can be shown that there exists a finite radius of convergence Raround the trivial solution of the original nonlinear FDE. The value of $R$is determined by the degree of nonlinearity of the governing operator, while the rate of convergence may vary significantly for different values of h. Therefore, the construction of an effective HAM formulation requires not only a suitable auxiliary operator but also a careful tuning of the parameter h.

Numerical and analytical investigations indicate that a wider stability interval is typically obtained when $0 < h \leq 1$, and the most effective convergence is often achieved for values in the ranges $h \in (0,1)$ or $h \in (0,0.5)$.

Sequence estimates further suggest that the zeroth-order approximation $u_0(t)$ already provides a reasonable estimate of the exact solution when the associated return maps generate a convergent sequence in a neighborhood of the trivial solution. In particular, the approximation becomes increasingly accurate as the iterates approach the exact solution.

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To assess the accuracy of the truncated series, error estimates are formulated in terms of the truncation index. These estimates show that the approximate solution satisfies the original nonlinear FDE up to a small residual term, and that the residual norm approaches zero for suitable choices of the convergence-control parameter h. This convergence analysis provides a rigorous theoretical foundation for the reliability and effectiveness of the HAM series solution in solving nonlinear fractional differential equations (H. Mohammed & I. Khlaif, 2014).

5. Illustrative Examples

The Homotopy Analysis Method (HAM) is employed to obtain analytical and semi-analytical solutions for a broad class of fractional-order differential equations subject to boundary conditions. The analysis considers two principal categories of problems: first-order fractional differential equations involving a zero-order fractional derivative and second-order fractional differential equations involving a first-order derivative. For problems in which exact closed-form solutions are unavailable, explicit approximate solutions are constructed in the form of convergent series expansions, which allows the convergence behaviour of the method to be examined in a systematic manner.

A comparison with existing approximate solutions demonstrates that the HAM approach provides a noticeable improvement in both accuracy and convergence rate (H. Mohammed & I. Khlaif, 2014). The resulting series representation is particularly advantageous for time-dependent fractional models, as it enables such problems to be reformulated and treated effectively as initial-value problems. In addition, the convergence-control parameter can be adjusted to achieve an appropriate balance between computational efficiency and convergence speed, thereby enhancing the practical performance of the method.

One of the major strengths of HAM is its ability to accommodate general-order fractional differential equations without requiring linearization or perturbation assumptions. By selecting suitable auxiliary linear operators and initial approximations, the method can produce highly accurate analytical solutions for several representative examples, including problems with variable coefficients and nonlinear source terms (Ahmad & Tauseef Mohyud-Din, 2014). This flexibility makes HAM a powerful tool for studying fractional models arising in physics, engineering, and applied mathematics.

The analysis of fractional differential equations with boundary conditions remains an active and important area of research. In particular, the qualitative properties of the chosen solution trajectories—such as monotonicity, linearity, continuity, and boundedness—play a crucial role in extending the validity of the series solution to broader classes of problems. These properties are also essential for establishing the stability and reliability of the obtained approximations.

Furthermore, the applicability of the proposed framework to nonlinear fractional boundary-value problems significantly broadens its scope. Convergence investigations show that the approximate solution remains uniformly bounded and preserves a positive lower bound independent of the truncation order of the series expansion. This observation provides additional theoretical support for the stability of the HAM solution and confirms the robustness of the method when applied to nonlinear fractional systems with boundary constraints.

6. Numerical Implementation and Computational Considerations

The implementation of the **Homotopy Analysis Method (HAM)** for fractional differential equations can be carried out through two complementary approaches, each offering a different balance between symbolic manipulation and numerical computation. The first is a **fully symbolic approach**, which allows the deformation equations and the corresponding series solution to be constructed directly and systematically for a given problem. The second is a more **computational or numerical approach**, which provides greater detail in the implementation process while reducing the dependence on advanced symbolic algebra. Together, these two approaches give HAM substantial flexibility, making it adaptable to a wide variety of fractional models and allowing symbolic software and numerical tools to be used effectively in tandem. Importantly, both formulations preserve the fundamental mathematical structure of the method, and each can be used to validate or support the other.

The series solution obtained from HAM may be truncated at any desired order, and the residual associated with the truncated approximation provides a useful measure of solution quality. This residual can be employed to assess convergence, guide the selection of the truncation level, and refine the choice of auxiliary parameters. In practice, the implementation is influenced by several key decisions, including the representation of the unknown function, the choice of the auxiliary linear operator, and the selection of the convergence-control parameter h. When the value of $h$becomes too large, the approximation may deteriorate, which provides an intuitive upper limit on the admissible range of this parameter. Although the computational time generally increases with the order of approximation because of the factorial growth in the number of terms appearing in the recurrence relations, the resulting polynomial degree remains significantly lower than that arising in many integer-order formulations, and the memory requirements are typically more manageable.

Throughout the procedure, careful attention must be paid to the evolution of the approximation and the behaviour of the residual. Both analytical checks and numerical diagnostics can be used to confirm the validity of the solution. Verification may also be achieved by comparing the HAM approximation with results obtained from other methods applied to the same problem, and when exact solutions are available, they provide a strong benchmark for accuracy assessment. In addition, the stability of the solution with respect to variations in $h$serves as an important indicator of reliability and demonstrates the rapid adaptability of the approximation as the convergence-control parameter is adjusted.

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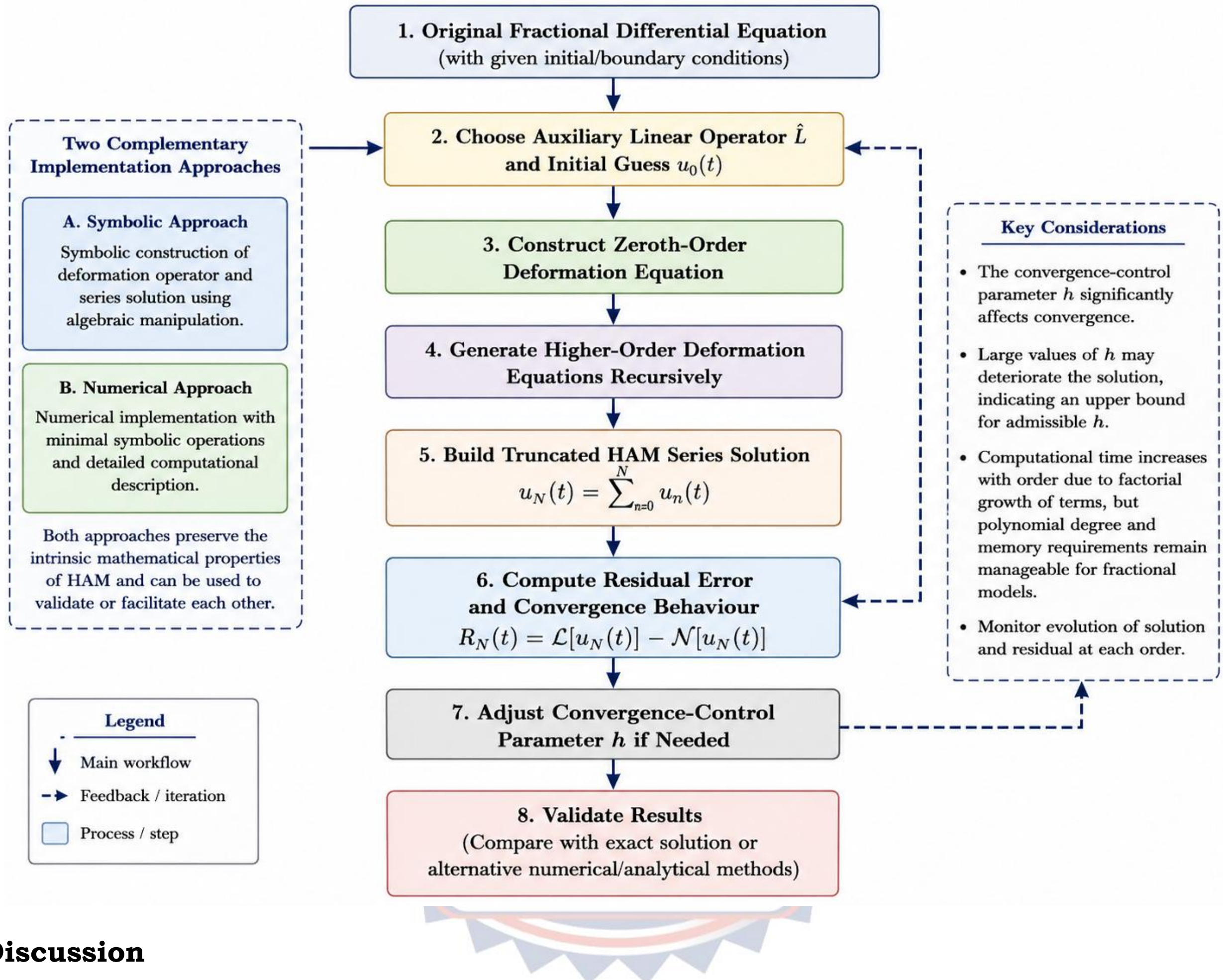
Figure 1 :Schematic representation of the HAM implementation and verification process for fractional differential equations, showing the transition from the original model to the truncated series solution and residual-based validation.



7. Discussion

The Homotopy Analysis Method (HAM) is a powerful analytical technique for constructing convergent series approximations to the solutions of a wide class of nonlinear problems. The method has been successfully applied to numerous nonlinear ordinary and partial differential equations, where it has demonstrated excellent flexibility, accuracy, and convergence properties. Unlike many traditional perturbation techniques, HAM does not require the presence of a small parameter and provides a systematic framework for controlling the convergence of the solution series through the introduction of auxiliary parameters and operators.

Although HAM has been extensively studied for integer-order differential equations, its application to fractional differential equations (FDEs) has only recently attracted significant attention. In the present work, a broad class of nonlinear fractional differential equations is analysed within the HAM framework, and the method is developed in a form that is suitable for fractional-order operators. This represents an important extension of the classical theory and provides a unified analytical framework for studying nonlinear systems with memory and hereditary effects.

Several recent studies have explored the use of HAM for different categories of fractional problems, highlighting the growing interest in this area and the need for a rigorous mathematical foundation. In particular, questions concerning the existence, uniqueness, convergence, and error analysis of HAM solutions for fractional models remain of fundamental importance. The formulation developed in this study addresses these issues by establishing theoretical conditions under which the homotopy series converges to a unique analytic solution and by providing criteria for estimating truncation errors and assessing the stability of the approximation.

From this perspective, the construction and analysis presented here may be regarded as a theoretical foundation for the application of HAM to nonlinear fractional differential equations. The framework not only justifies the validity of the method from a mathematical standpoint but also significantly broadens the range of problems to which it can be applied. Furthermore, the obtained results are not restricted to the specific models considered in this study; they are applicable to a much wider class of nonlinear fractional equations than those treated in earlier investigations. Consequently, the proposed theory possesses substantial scientific value and offers promising opportunities for future research in fractional calculus, nonlinear dynamics, and applied mathematical modeling.

8. Conclusion

Fractional differential equations (FDEs), which play a crucial role in fields such as physics, engineering, finance, biology, and control theory, have been investigated in this study through the homogeneous formulation of the Homotopy Analysis Method (HAM). Because fractional-order derivatives are less extensively studied from both theoretical and analytical perspectives than classical integer-order derivatives, the development of reliable analytical techniques for solving FDEs remains an important research challenge. In this context, HAM provides a flexible and powerful framework for constructing convergent series solutions without requiring linearization or perturbation assumptions.

Both the theoretical foundations and the computational implementation of the method have been examined in detail. A rigorous framework has been established for the existence, uniqueness, and convergence of the HAM series solution, providing a solid mathematical basis for the proposed approach. The convergence analysis, together with residual-based error assessment, demonstrates that the method can produce accurate approximate solutions for a broad class of nonlinear fractional differential equations while maintaining computational efficiency.

Fractional-order models possess distinctive characteristics, such as memory and hereditary effects, that make them particularly suitable for describing complex dynamical systems. However, these same features often render analytical treatment difficult, and many conventional methods fail to provide satisfactory solutions. The present work highlights the effectiveness of HAM as a practical analytical tool for such problems and provides a systematic procedure for selecting suitable auxiliary parameters, linear operators, and convergence-control ranges to ensure stable and accurate computations (Ahmad & Tauseef Mohyud-Din, 2014).

The obtained results indicate that HAM is capable of handling both weakly and strongly nonlinear fractional systems and can be readily extended to more advanced models involving multi-term, higher-order, variable-order, and coupled fractional differential equations. Owing to its flexibility, strong convergence characteristics, and relatively low computational cost, the method offers significant potential for the analysis of complex fractional models encountered in real-world applications.

Future research may focus on the development of adaptive convergence-control strategies, the integration of HAM with modern numerical and data-driven techniques, and the extension of the framework to distributed-order and stochastic fractional systems. Such developments would further enhance the applicability of HAM and contribute to the advancement of analytical methods for fractional-order modeling and simulation.

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